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Embedded Markov Chain of an M/G/1 Retrial Queue

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Abstract : Single server retrial queueing system with Poisson arrival and retrial and generally distributed service time is considered. The stochastic process $\{N_i\}$ of the number of customers in the orbit at the departure epochs is studied. The Markov property of the process is verified. One step transition probabilities of the Markov chain $\{N_i\}$ are computed.The ergodicity of the Markov Chain has been verified.

Key words* - *Retrial Queue, Orbit, Markov Chain, Ergodicity

Introduction

Retrial queueing system is characterized by the following feature: a customer arriving when all servers accessible for him are busy leaves the service area, but after some random time repeats his demand. This feature has a special role in several computer and communication networks.

Consider a single server queueing system in which customers arrive in a Poisson process with rate λ. These customers are identified as primary customers. If the server is idle at the arrival of a primary customer, the arriving customer begins to be served immediately and leaves the system after service completion. Otherwise, if the server is busy, the arriving customer leaves the service area and joins a pool of unsatisfied customers called orbit. Every such customers produces a Poisson process with intensity μ. The service time distribution function is B(x) for both primary and retrial customers.

At time t, let N(t) be the number of customers in the orbit and C(t) be the server status.

ie: $C(t) = \begin{cases} 1, if\ the\ server\ is\ busy \\ 0, if\ the\ server\ is\ idle \end{cases}$

The process (C(t), N(t)) describes the number of customers in the system in the simplest way. This is the most important process associated with this queueing system. But, when the service time distribution is *not* exponential, the process is not Markov.

So, we study the system using the embedded Markov chain of the number of orbital customers at the departure epochs.

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Model Discription

The queueing process evolves in the following manner. Suppose that the $(i-1)^{th}$ customer completes his service at the epoch t_{i-1} and the server becomes free. Even if there are some customers in the system who wants to get service, they cannot occupy the server immediately, because of their ignorance of the server state. So the next, i^{th} customer, enters the server only after some time interval R_i during which the server is idle. If the number of orbital customers at the epoch t_{i-1} is n, the random variable R_i has an exponential distribution with parameter $\lambda + n\mu$. The i^{th} customer is a primary customer with probability $\frac{\lambda}{\lambda+n\mu}$ and it is an orbital customer with probability $\frac{n\mu}{\lambda+n\mu}$. At the epoch $\xi_i = t_{i-1} + R_i$, the i^{th} customer starts his service and continues during a time S_i. All primary customers arrived during the service time joins the orbit. (The retrials arrived during this time do not influence the system) At the epoch, $t_i = \xi_i + S_i$ the i^{th}customer completes his service and the server becomes free again.

Let $N_i = N(t_i)$ be the number of customers in the orbit at the i^{th} departure epoch t_i. Then,

$$N_i = N_{i-1} - B_i + \nu_i$$

Where, $B_i = \begin{cases} 0, \; if\ the\ i^{th} customer\ is\ a\ primary\ customer \\ 1, \; if\ the\ i^{th} customer\ is\ a\ retrial\ customer \end{cases}$

and, ν_i = # of primary customers arrived during the service time of the i^{th} customer.

B_i is a Bernoulli's r.v with,

$\Pr\{(B_i = 0)/(N_{i-1} = n)\} = \lambda/(\lambda + n\mu)$ and,

$\Pr\{(B_i = 1)/(N_{i-1} = n)\} = n\mu/(\lambda + n\mu)$.

The rv $\nu_i$doesn't depend on events which occurred before ξ_i and it's distribution is,

$k_n = \Pr\{\nu_i = n\} = \int_0^\infty \Pr\{(\nu_i = n)/(S_i = t)\}\, d\,B(t) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\,B(t)$

It's generating function is, $k(z) = \sum_{n=0}^{\infty} k_n z^n = \sum_{n=0}^{\infty} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\,B(t)\; z^n$

$= \int_0^\infty e^{-\lambda t} \left(\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} z^n\right) d\,B(t) = \int_0^\infty e^{-\lambda t} e^{\lambda t z}\, d\,B(t) = \int_0^\infty e^{-(\lambda - \lambda z)t}\, d\,B(t)$

$= \beta(\lambda - \lambda z)$, where, $\beta(s) = \int_0^\infty e^{-st}\, d\,B(t)$ is the Laplace- Stieljes transform of the service time distribution $B(x)$.

Again, $E(\nu_i) = \sum_{n=1}^{\infty} nk_n = k'(1) = \beta'(0)(-\lambda) = \left(-E(S_i)\right)(-\lambda) = \lambda E(S_i) = \rho$

From these observations it is clear that the random variable N_i does not depend on the history before N_{i-1}. So $\{N_i\}$ is a Markov chain with state space Z_+..

One Step Transition Probability

The one-step transition probabilities of the Markov chain $\{N_i\}$ is given by,

$$p_{mn} = \Pr\{(N_i = n)/(N_{i-1} = m)\}$$

Clearly $p_{mn} \neq 0$, for $m = 0,1,2, \ldots\ldots n+1$.

Now, $p_{mn} = \Pr\{(N_i = n)/(N_{i-1} = m)\} = \Pr\{(N_{i-1} - B_i + \nu_i = n)/(N_{i-1} = m)\}$

$$= \Pr\{(\nu_i = n - m + B_i)/(N_{i-1} = m)\}$$

$= \Pr\{(\nu_i = n - m + 0)/(N_{i-1} = m)\}\text{x}\Pr\{B_i = 0\} + \Pr\{(\nu_i = n - m + 1)/(N_{i-1} = m)\}\text{x}\Pr Bi{=}1$

$= k_{n-m}\left(\frac{\lambda}{\lambda + m\mu}\right) + k_{n-m+1}\left(\frac{m\mu}{\lambda + m\mu}\right).$

Ergodicity

Theorem: The embedded Markov chain $\{N_i\}$ is ergodic iff $\rho < 1$.

Proof: Because of the recursive structure of the distribution of $\{N_i\}$ we can use the criteria using *mean drift* or *Liyapunov functions*. So the proof follows from the following statement:

Statement 1: For an irreducible and aperiodic Markov chain ξ_i with state space S, a sufficient condition for ergodicity is the existence of a non negative function $f(s), s\epsilon S$ and $\varepsilon > 0$ such that the mean drift

$$x_s \equiv E(f(\xi_{i+1}) - f(\xi_i)/\xi_i = s)$$

is finite for all $s\epsilon S$ and $x_s \leq -\varepsilon$ for all $s\epsilon S$ except perhaps a finite number.

In the simple case when the state space $Z_+$of non negative integers it is usually sufficient to consider the function $f(n) = n$. It means that a chain is ergodic if its mean drift

$$x_n \equiv E(\xi_{i+1} - \xi_i)/\xi_i = n)$$

Is less than some negative number $-\varepsilon$ for all $n \geq N$, where N is sufficiently large integer.
Of course, if there exists the limit

$$x = \lim_{n \to \infty} x_n$$

This condition holds iff $x < 0$.
For the Markov Chain under consideration the mean drift introduced above is,

$$x_n = E(N_{i+1}/N_i = n)$$

$$= E(-B_{i+1} + \nu_{i+1} = n)/N_i = n)$$

$$= E(-B_{i+1}/N_i = n) + E(\nu_{i+1}/N_i = n)$$

$$= P(-B_{i+1} = 1/\ N_i = n) + E(\nu_{i+1})$$

$$= -\frac{n\mu}{\lambda + n\mu} + \rho$$

So that $\lim_{n\to\infty} x_n = -1 + \rho$. This limit is negative iff $\rho < 1$.

Applying the above Foster's criterion we can say that for $\rho < 1$ the embedded Markov chain is ergodic.

Conclusion

M/M/1 retrial queues can be solved easily by matrix geometric method. But, when the service time distribution is not exponential the solution becomes rather difficult. An analytic solution of M/G/1 Retrial Queue has given in Retrial Queues by Fallin and Templeton.

References .

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